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Effects of Coriolis forces on the Kelvin-Helmholtz instability in a stratified plasma

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Abstract : In this note the instability in the Kelvin-Helmholtz configuration of a stratified plasma has been studied to include the joint influence of effects of F L R. and Coriolis forces The plasma is assumed to be moving with a uniform streaming velocity and the axis of rotation is taken along the direction of the horizontal magnetic field. The dispersion relation has been solved numerically and it is found that the Coriolis forces have dual role stabilizing for small wave numbers and destabilizing for large wave numbers The plasma can in fact be stabilized by these effects over a certain small wave number range over which it is unstable in their absence.

1. Introduction

The stability of the well known Kelvin-Helmholtz configuration has been studied by several authors and a detailed account of these investigations has been given by Chandrasekhar (1961) The importance and the stabilizing influence of FLR effects on plasma instabilities has been pointed out by Rosenbluth *et al* (1962); Roberts and Taylor (1962) and Jukes (1964)

Henning, Melchior and Popovich (1968) have studied the FLR effects on the Kelvin-Helmholtz instability of a fully ionized plasma in which there is a one-dimensional stratification in the density in a direction transverse to that of the magnetic field. But in this study the effects of the Coriolis forces were not included. In view of the importance of the effect of Coriolis forces, Bhatia (1970) has studied the FLR effects on the Kelvin-Helmholtz instability of a rotating magnetised plasma of variable density In his study the axis of rotation was taken to be in a direction transverse to that of the static uniform horizontal magnetic field. In actual physical situations, whether astrophysical or geophysical, the case when the system is partaking in a uniform rotation about the direction of magnetic field is also of interest.

It would, therefore, be of importance and equally interesting to examine the influence of the simultaneous presence of the effects of FLR and Coriolis

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forces on the K-H instability in a stratified plasma which is rotating uniformly about the direction of the magnetic field. This aspect forms the subject matter of this note where we have considered the case of wave propagation transverse to that static horizontal magnetic field. In the absence of streaming motion, a similar study (in the presence of effects of viscosity) has been carried out by Bhatia and Steiner (1975).

2. Perturbation equations

Consider a plasma configuration which is in equilibrium under the action of a uniform horizontal magnetic field $\boldsymbol{H} = (H, 0, 0)$, and which is in uniform rotation about the x-axis with an angular velocity Ω i.e., $\boldsymbol{\Omega} = (\Omega, 0, 0)$. The plasma is assumed to be moving with a uniform streaming velocity $\boldsymbol{U} = (0, U, 0)$. The finiteness of the ion Larmor radius (FLR) is taken into account by considering the pressure as a tensor P_{ij} ($\boldsymbol{P}$) in the hydromagnetic equations. Let $\boldsymbol{u}(u, v, w)$, $\boldsymbol{h}(h_x, h_y, h_z)$, $\delta\rho$, $\delta\boldsymbol{P}$ respectively denote the perturbation in velocity, magnetic field $\boldsymbol{H}$, density ρ and stress tensor $\boldsymbol{P}$. The linearized perturbation equations appropriate to the problem are

$$\rho \frac{\partial \boldsymbol{u}}{\partial t} + \rho(\boldsymbol{U}\cdot\nabla)\boldsymbol{u} = -\nabla\delta\boldsymbol{P} + \frac{1}{4\pi}(\nabla \times \boldsymbol{h}) \times \boldsymbol{H} + \boldsymbol{g}\delta\rho + 2\rho(\boldsymbol{u} \times \boldsymbol{\Omega}), \tag{1}$$

$$\frac{\partial}{\partial t}\delta\rho + (\boldsymbol{U}\cdot\Delta)\delta\rho + (\boldsymbol{u}\cdot\nabla)\rho = 0, \tag{2}$$

$$\frac{\partial \boldsymbol{h}}{\partial t} + (\boldsymbol{U}\cdot\nabla)\boldsymbol{h} = (\boldsymbol{H}\cdot\nabla)\boldsymbol{v}, \quad \nabla\cdot\boldsymbol{h} = 0, \tag{3}$$

$$\nabla\cdot\boldsymbol{v} = 0, \tag{4}$$

where $\boldsymbol{g} = (0, 0, -g)$ is acceleration due to gravity. For the horizontal magnetic field $\boldsymbol{H}(H, 0, 0)$, the stress tensor P_{ij}, taking into account the finite ion gyration radius, has the components

$$
\begin{aligned}
P_{xx} &= p, \\
P_{yy} &= p - \rho\nu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right), \\
P_{zz} &= p + \rho\nu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right), \\
P_{xy} &= P_{yx} = -2\rho\nu\frac{\partial u}{\partial z}, \\
P_{zx} &= P_{xz} = 2\rho\nu\frac{\partial u}{\partial y}, \\
P_{yz} &= P_{zy} = \rho\nu\left(\frac{\partial v}{\partial y} - \frac{\partial w}{\partial z}\right)
\end{aligned}
\tag{5}
$$

Here p is the scalar part of pressure and $\rho\nu = NT/4\omega_H$, where ω_H is the ion gyration frequency, while N and T are respectively the number density and ion temperature.

We seek solutions of the above equations, by analysing them in terms of normal modes, of the form

$$\exp(ik_y y + int), \tag{6}$$

where k_y is the wave number of the perturbation along y-axis and n is the growth rate of the perturbations We are thus considering here the transverse mode of wave propagation, as in Bhatia (1970) and Bhatia and Steiner (1975), as this mode is easily tractable mathematically Further it is hoped that, is an earlier studies, the essential features of the instability are revealed by these transverse waves and that the study gives us at least an insight into the tendencies of the influences of the physical effects studied

The influence of the effects of Hall currents should also be of importance in the stability problem studied here However for the horizontal magnetic field Hall current term does not contribute any effect for the mode considered (Eq (6)) The Hall effect thus does not have any influence on the stability of the considered plasma

For the perturbations of the form (6), equations (1) and (4), become, on using (2), (3) and (5) and writing the operator $D = d/dz$,

$$i\rho(n+k_yU)u = -2ik_y\nu uD\rho, \tag{7}$$

$$i\rho(n+k_yU)v = -ik_y\delta p - \nu(ik_yv - Dw)D\rho + \rho\nu(D^2-k_y^2)w + 2\rho\,\Omega_w, \tag{8}$$

$$i\rho(n+k_yU)w$$

$$= -D\delta p - \nu(D\rho)Dv - \rho\nu(D^2-k_y^2)v - ik_y\nu(D\rho)w - \frac{ig(D\rho)w}{(n+k_yU)} - 2\rho\Omega_v, \tag{9}$$

and

$$ik_yv + Dw = 0 \tag{10}$$

Eliminating δp between (8) and (9) and finally putting v in terms of w from (10) we get the differential equation in w as

$$(n+k_yU)[D(\rho Dw) - \rho k_y^2 w] + 2k_y\,\Omega\, wD\rho$$

$$+2k_y\nu[D\{(Dw)(D\rho)\} - k_y^2(D\rho)w] - \frac{gk_y^2(D\rho)w}{n+k_yU} = 0 \tag{11}$$

In obtaining (11), it has assumed that the stratification in density is one-dimensional and is along the z-direction.

3. The dispersion relation

In the following we consider a plasma in which density ρ varies exponential upwards i.e.,

$$\left.\begin{aligned}\rho &= \rho_1 \exp(\beta z), \quad 0 \leqslant z \leqslant d, \\ &= 0, \quad \text{elsewhere}\end{aligned}\right\} \tag{12}$$

where ρ_1 and β are constants. It is assumed that the plasma is confined between two rigid boundaries, at $z = 0$ and $z = d$, and is infinitely extending along the two horizontal directions. Hence on these boundaries w must be zero.

For density distribution of the form (12), eq. (11) in w becomes

$$D^2w+\beta Dw-k_y^2\left[\frac{(n+k_yU)+2\nu\beta k_y-\dfrac{2}{k_y}\Omega\beta+\dfrac{g\beta}{n+k_yU}}{(n+k_yU)+2\nu\beta k_y}\right]w=0. \tag{13}$$

Appropriate to the boundary condition at the lower boundary $z = 0$, the solution of (13) can be written as

$$w(z) = A(e^{m_1z}-e^{m_2z}), \tag{14}$$

where m_1 and m_2 are roots of the equation

$$m^2+\beta m-k_y^2\left\{n+k_yU+2\nu\beta k_y-\frac{2\beta\Omega}{k_y}+\frac{g\beta}{n+k_yU}\right\}\times[n+k_yU+2\nu\beta k_y]^{-1}=0. \tag{15}$$

Since w must vanish at $z = d$, we must have

$$\exp(m_1-m_2)d = 1$$

or

$$(m_1-m_2)d = 2iS\pi, \tag{16}$$

S being an integer.

Putting the values of the roots m_1 and m_2 from equation (15) in (16) we get the dispersion relation as

$$n^2+2n\left(U+\nu\beta-\frac{\beta\Omega}{\alpha^2+k_y^2}\right)k_y+\left\{2\nu\beta k_y^2U+k_y^2U^2+\frac{k_y^2\beta(g-2\Omega U)}{\alpha^2+k_y^2}\right\}=0, \tag{17}$$

where

$$\alpha^2 = \frac{\pi^2S^2}{d^2}+\frac{1}{4}\beta^2 \tag{18}$$

Substituting $\bar{n} = \frac{n}{\alpha}$, $\bar{k} = \frac{k_y}{\alpha}$ and $\bar{\nu} = \nu\beta$, the non-dimensional form of dispersion relation is

$$\bar{n}^2 + 2\bar{n}\bar{k}\left(U + \bar{\nu} - \frac{A}{1+\bar{k}^2}\right) + \bar{k}^2\left\{U^2 + 2U\bar{\nu} + \frac{1}{1+\bar{k}^2}(B - 2AU)\right\} = 0, \tag{19}$$

where

$$A = \frac{\Omega\beta}{\alpha^2}, \quad \text{and} \quad B = \frac{g\beta}{\alpha^2} \tag{20}$$

Here A and B are pure numbers and measure the effects respectively of coriolis and gravitational forces

The solution of equation (19) is

$$\bar{n} = \bar{k}\left[-\left(\left\{\bar{\nu} - \frac{A}{1+\bar{k}^2}\right\} + U\right) \pm \left\{\left(\bar{\nu} - \frac{A}{1+\bar{k}^2}\right)^2 - \frac{B}{1+\bar{k}^2}\right\}^{1/2}\right] \tag{21}$$

Here two cases arises : (i) $\beta < 0$ and (ii) $\beta > 0$

Case (i) : $\beta > 0$ (Stable Stratification)

It can be easily seen from (21) that when $\beta < 0$, both A and B are negative and both the roots of $\bar{n}$ are then real. Hence the considered plasma is always stable

Case (ii) . $\beta > 0$ (Unstable Stratification)

It is clear from equation (21) that when $\bar{\nu} = A/(1+\bar{k}^2)$ (including $\bar{\nu} = A = 0$) the system is unstable for all k. However the system is stablized by these effects for wave numbers $0 < k < k_c$, where

$$k_c = \left[\frac{-\{2\bar{\nu}(\bar{\nu} - A) + B\} \pm \sqrt{B^2 - 2\bar{\nu}B(\bar{\nu} + A)}}{2\bar{\nu}^2}\right]^{1/2}$$

The system is of course unstable for all $k > k_c$

A few numerical calculations of k_c for different values of A have been carried and are given in Table 1 It is found that k increases with A Thus the range of k_c, over which the plasma is stablized, increases with Coriolis forces The effect of Coriolis forces is therefore, stabilizing.

Table 1. Value of k_c against A

For $B = 100$, $\nu = 1{\cdot}0$

A	10·0	15·0	20·0	25·0	30·0
k_c	1·78	2·28	2·69	2·94	3·13

Table 2 Value of growth rate (Imag ($\bar{n}$)) against wave number ($\bar{k}$) for $U = 1\ 0$ and $B = 10\ 0$

Sl. No	Parameters	($\bar{k}$)										
		0·0	0·2	0·4	0 6	0 8	1 0	1 2	1 4	1 6	1·8	2·0
1.	$\bar{\nu} = 2·0$ $A = 2·0$	0·0	0·6199	1·1692	1 5956	1 8741	1 9999	1·9736	1·7844	1·3769	0·2717	0·0
2	$\bar{\nu} = 2·0$ $A = 3·0$	0·0	0·5944	1·1507	1 6222	1 9707	2·1794	2 2465	2·1712	1·9396	1 4930	0·4000
3.	$\bar{\nu} = 2\ 0$ $A = 4·0$	0·0	0·4982	1 0216	1·5258	1·9439	2·2360	2 3904	2 4076	2·2857	2·0060	1·4966
4	$\bar{\nu} = 2·0$ $A = 5·0$	0 0	0·2632	0·7247	1·1787	1·7884	2·1794	2 4286	2 5361	2·5066	2 3364	2 0000
5.	$\bar{\nu} = 0·5$ $A = 2·0$	0·0	0·5510	1·0674	1·5191	1·8897	2·1794	2 3988	2·5614	2 6797	2 7638	2 8213
6	$\bar{\nu} = 1\ 0$ $A = 2·0$	0·0	0·5920	1·1381	1 6022	1·9676	2·2360	2 4190	2 5328	2·5883	2 5956	2·5612
7.	$\bar{\nu} = 1·5$ $A = 2·0$	0·0	0·6143	1 1710	1·6268	1·9626	2 1794	2·2880	2 2999	2·2220	2·0531	1 7776
8	$\bar{\nu} = 2·0$ $A = 2·0$	0·0	0·6199	1·1692	1·5956	1·8741	1 9999	1 9736	1 7844	1·3769	0·2717	0 0
9.	$\bar{\nu} = 3·0$ $A = 2\ 0$	0·0	0·5816	1·0580	1·3442	1·3693	1·0000	0 0	0 0	0·0	0·0	0 0
10	$\bar{\nu} = 4\ 0$ $A = 2\ 0$	0·0	0 4605	0·7423	0 6122	0 0	0·0	0·0	0·0	0 0	0·0	0 0

In order to estimate the influence of the various physical parameters on the growth rate of the unstable mode of propagation, numerical calculations were performed to obtain the roots of equation (19) for several values of the parameters $\bar{\nu}$, A, U, keeping $B = 10{\cdot}0$ and $U = 1{\cdot}0$ fixed. These calculations are presented in Table 2 where we have given the values of the growth rate (Imaginary part of $\bar{n}$) against wave number $\bar{k}$ for $A = 2{\cdot}0, 3{\cdot}0, 4{\cdot}0, 5{\cdot}0$; $\bar{\nu} = 0{\cdot}5, 1{\cdot}0, 1{\cdot}5, 2{\cdot}0, 3{\cdot}0, 4{\cdot}0$. It is seen that for small values of the wave numbers the growth rate of the unstable mode decreases on increasing the Coriolis parameter A for same values of $\bar{k}$ and for the same values of the other parameters However, for large wave numbers the growth rate is found to increases with increasing values of the parameter A. Thus we find that the Coriolis forces have a dual role, stabilizing for waves of large wavelength (or small wave numbers) and destabilizing for waves of small wave length (or large wave numbers). Also it is found that the range of wave numbers, over which the Coriolis forces have a stabilizing influence, increases with increase in the values of the Coriolis force parameter.

The influence of the FLR effect is also found to be stabilizing for the small range of wave numbers over which the FLR can possibly have destabilizing influence is also found to decrease with increasing rotation frequency and on increasing FLR These results are in agreement with the earlier observations of a streaming plasma (Bhatia 1970) when $\mathbf{\Omega} \perp \boldsymbol{H}$ and non-streaming plasma (Bhatia and Steiner 1975), when $\mathbf{\Omega} \parallel \boldsymbol{H}$.

It may be added that the analysis is breaks down for very small values of $\bar{k}$, as this implies long wave length disturbances and $\lambda\,\Omega$ (λ = wavelength) will exceed the speed of sound and/or the Coriolis force will cease to be small perturbation.

We may thus conclude that the streaming plasma is certainly stabilized over a certain wave number range of perturbations (by Coriolis forces and FLR effects) over which it is unstable in their absence.

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